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ON S_{mBE}-ALGEBRAS

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Abstract

In this article, class of all bounded commutative self distributive modal BE-algebras S_{mBE} is introduced. Finally, we show that the class of all bounded commutative self distributive modal BE-algebras, S_{mBE} , coincides with the class of all topological Heyting algebras.

Keywords: ((modal) Heyting,(positive) implication) algebra,(modal) BE-algebra.

1. Introduction and Preliminaries

Arend Heyting, a student of Brouwer's, formalized the mathematical philosophy of intuitionism into his namesake algebras. Pseudo–Boolean algebras characterize algebraically intuitionistic logic. These algebras, called also Heyting algebras or pseudo complemented lattices, are dual to Brouwerian algebras, which have been investigated in detail by Mckinsey and Tarski. The Heyting algebras are most simply defined as a certain type of lattice. Algebras including Heyting algebras, have played an important role and have its comprehensive applications in many aspects including the genetic code of biology and dynamical systems. H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra as a generalization of a dual BCK-algebra [3]. A. Rezaei and et al. get some results on BE-algebras and introduced the notion of commutative ideals in BE-algebras and proved several characterizations of such ideals [9, 10]. A. Walendziak, investigate the relationship between BE–algebras, implication algebras, and J-algebras. Moreover, he defined commutative BE-algebras and state that these algebras are equivalent to the commutative dual BCK-algebras [13]. Also, A. R. Hadipour and et al. proved that every Heyting algebra is a BE-algebra and every bounded commutative self-distributive BE-algebra is a Heyting algebra [4].

The algebraic approach to logic, begun by Boole and his followers (W. Stanley Jevons, Peirce, Schroder), has recently been taken up and strongly developed in connection with the formulation of non-classical logics: many-valued logics of Lukasiewicz and Post, intuitionistic logic, modal logic, positive logic, constructive logic with strong negation and others.

In this paper, we construct a class of all bounded, commutative, self-distributive, modal BE–algebras, (S_{mBE} for briefly) in Rasiow's style. We investigate the relation between modal Heyting algebras with modal BE-algebras. In fact, the purpose of this paper is to show that the class of all bounded commutative self-distributive modal BE-algebras coincides with the class of all topological Heyting algebras.

In the following we review some basic definitions and it also establishes notational conventions. Also, assume that the notion of propositional language L is defined as usual.

Definition 1. [3] By a BE-algebra we shall mean an algebra $(X;*,1)$ of type $(2,0)$ satisfying the following axioms:

(BE1) $x*x=1$,
(BE2) $x*1=1$,

(BE3) $1*x = x,$

(BE4) $x*(y*z) = y*(x*z),$

for all $x, y, z \in X$.

A relation "$\leq$" on X is defined by $x \leq y$ if and only if $x*y = 1$. In what follows, let X be a BE–algebra unless otherwise specified.

A BE-algebra X is said to be self distributive if $x*(y*z) = (x*y)*(x*z)$, for all $x, y, z \in X$.

We say that X is commutative if $(x*y)*y = (y*x)*x$, for all $x, y, z \in X$.

We note that "$\leq$" is reflexive by (BE1). If X is self distributive, then relation "$\leq$" is a transitive order on X , because if $x \leq y$ and $y \leq z$, then

$$x*z = 1*(x*z) = (x*y)*(x*z) = x*(y*z) = x*1 = 1.$$

Hence $x \leq z$. If X is commutative, $x \leq y$ and $y \leq x$, then

$$x = 1*x = (y*x)*x = (x*y)*y = 1*y = y.$$

So, relation "$\leq$" is antisymmetric. Thus, if X is a commutative self distributive BE-algebra, then relation "$\leq$" is a partial order set on X .

X is called bounded if there exists the smallest element 0 of X i.e. $0*x = 1$, for all $x \in X$. Given a bounded BE-algebra X with 0 as the smallest element, we denote $x*0$ by Nx , then N can be regarded as a unary operation on X . If $NNx = x$, then x is called an involution of X . A bounded BE–algebra X is called involutory if any element of X is involution.([2])

Example 1. [11] (i). Let $X = \{1, a, b, c, d\}$ be a set with the following table:

*	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	d
b	1	c	1	c	d
c	1	b	b	1	d
d	1	a	b	c	1

Then $(X;*,1)$ is a self distributive BE-algebra.

(ii). Let $X = \{1, a, b, c\}$ be a set with the following table:

*	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Then $(X;*,1)$ is a commutative BE-algebra.

(iii). Let $\mathbb{N}$ be the set of all natural numbers and "$*$" be the binary operation on $\mathbb{N}$ defined by:

$$x*y = \begin{cases} y & if \quad x = 1 \\ 1 & if \quad x \neq 1 \end{cases}$$

Then $(\mathbb{N};*,1)$ is a non-commutative BE-algebra.

(iv). Let $X = \{1, a, b, c\}$ be a set with the following table:

*	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	1	1	1
0	0	a	b	1

Then $(X;*,0,1)$ is an involutory BE-algebra.

Definition 3. [1] An implication algebra is a set X with a binary operation "$\rightarrow$" which satisfies the following axioms:

(I1) $(x \rightarrow y) \rightarrow x = x,$

(I2) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x,$

(I3) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z),$

for all $x, y, z \in X$.

Definition 4. [8] An abstract algebra $(X;\to,1)$ of type $(2,0)$ is said to be a positive implication algebra if satisfies in the following axioms:

(PI1) $x\to(y\to x)=1,$

(PI2) $((x\to(y\to z))\to((x\to y)\to(x\to z))=1,$

(PI3) if $x\to y=1,$ and $y\to x=1,$ then $x=y,$

(PI4) $x\to 1=1,$

for all $x,y,z\in X$.

Definition 5. [8] An algebra $(X;\to,\vee,\wedge,1)$ is a Heyting algebra if and only if $(X;\to,1)$ is a positive implication algebra and moreover the following conditions aresatisfied:

(PI5) $x\to(x\vee y)=1,$

(PI6) $y\to(x\vee y)=1,$

(PI7) $(x\to z)\to((y\to z)\to((x\vee y)\to z))=1,$

(PI8) $(x\wedge y)\to x=1,$

(PI9) $(x\wedge y)\to y=1,$

(PI10) $(x\to y)\to((x\to z)\to(x\to(y\wedge z)))=1,$

for all $x,y,z\in X$.

Example 2. [14] (i). Every bounded chain lattice L is a Heyting algebra.

(ii). Every Boolean algebra is a Heyting algebra and every Heyting algebra is a distributive lattice.

Now, let X be a commutative self-distributive BE-algebra with 0 as the smallest element. Suppose that $\neg x=x*0$.

Theorem 1. [4] If $(X;\to,\vee,\wedge,0,1)$ is a Heyting algebra, then $(X;\to,1)$ is a BE-algebra.

The following example shows that the converse of Theorem 1 may not necessarily hold.

Example 3. [4] Let $\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and "$*$" be a binary operation on $\mathbb{N}_0$ defined by:

$$x*y=\begin{cases}0 & if \quad y\le x\\ y-x & if \quad x<y\end{cases}$$

Then $(\mathbb{N}_0;*,0)$ is a commutative BE-algebra, but it is not a Heyting algebra, because it is not bounded.

Theorem 2. [4] Let $(X;*,0,1)$ be a bounded commutative self distributive BE-algebra. Then $(X;\to,\vee,\wedge,0,1)$ is a Heyting algebra.

The modal logic discussed in the present article corresponds to the system S4 of Lewis and Langford.

It was originated as a result of a philosophical criticism concerning classical (material) implication.

Definition 6. [15] An algebra $(X;*,\square,1)$ of type $(2,1,0)$ is called a modal BE-algebra if it satisfies the following:

(BE) $(X;*,1)$ is a BE-algebra,

(MBE1) $\square 1=1,$

(MBE2) $\square x\le x,$

(MBE3) $\square x=\square\square x,$

(MBE4) $\square(x*y)=\square x*\square y,$

for all $x,y\in X.$

In the following we state some preliminaries [8].

Example 4. (1). Let $X=\{1,a,b,c\}$. Define the operations "$*$" and "$\square$" on X as follow:

*	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	a	1	c
c	1	b	1	1

And

*	1	a	b	c
□	1	a	c	c

Then $(X;*,\square,1)$ is a modal BE-algebra.

Definition 7. [8] Let $S=(L,C_L)$ where, L is a language and C_L is a consequence operation in L . The class S , is the class of standard systems of implicative extensional propositional calculi. A system $S=(L,C_L)$ of propositional calculus in S will be said to be consistent provided there exist a formula α of L such that $\alpha \notin C_L(\phi)$.

Consider two languages $L=(A,F)$, $L'=(A',F')$ where, $A=(V,L_0,L_1,L_2,U)$ and $A'=(V',L'_0,L'_1,L'_2,U')$ also F, F' are the set of all formulas over the set A, A', respectively. The sets L_0, L'_0 are of all constants, the sets L_1, L'_1 are of all unary connectives, the sets L_2, L'_2 are of all binary connectives, finally elements in U will be called auxiliary signs.

Definition 8. [8] The languages L and L' will be said to be similar provided that $L_i = L'_i$ for $i=0,1,2$. The language L' is an extension of the language L provided $V \subseteq V'$ and $L_i = L'_i$, for $i=0,1,2$.

Definition 9. [8] Two consistent systems $S=(L,C_L)$ and $S'=(L',C'_L)$ in S where L and L' are similar languages will be said to be logically equivalent provided that the class of all S-algebras coincides with the class of all S'-algebra.

In fact, any class L of consistent logically equivalent systems in S determines a "Logic". For instance, the class of all logically equivalent systems S in S for which the class of S-algebras coincides with the class of pseudo Boolean algebras determines intuitionistic logic.

2. Relationship between S_{mBE}-algebras and topological Heyting algebras

In this section we shall deal with the propositional calculus $S_{mBE}=(L,C_L)$ defined as follows:

The alphabet $A=(V,L_0,L_1,L_2,U)$ of language L is as follows: V is the set of all propositional variables, L_0 is the set of all constants, L_1 contains unary connective $\Box$, L_2 contains binary connective $\Rightarrow, \wedge, \vee$.

The set U of logical axioms consists of all formulas of the form:

(A1) $\alpha \Rightarrow (\beta \Rightarrow \alpha)$,
(A2) $(\alpha \Rightarrow (\beta \Rightarrow \gamma))$
$\Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma))$,
(A3) $\alpha \Rightarrow (\alpha \vee \beta)$,
(A4) $\beta \Rightarrow (\alpha \vee \beta)$,
(A5) $((\alpha \Rightarrow \gamma) \Rightarrow ((\beta \Rightarrow \gamma) \Rightarrow ((\alpha \vee \beta) \Rightarrow \gamma))$,
(A6) $(\alpha \wedge \beta) \Rightarrow \alpha$,
(A7) $(\alpha \wedge \beta) \Rightarrow \beta$,
(A8) $(\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma)$
$\Rightarrow (\alpha \Rightarrow (\beta \wedge \gamma))$,
(A9) $\Box\alpha \wedge \Box\beta \Rightarrow \Box(\alpha \wedge \beta)$,
(A10) $\Box\alpha \Rightarrow \alpha$,
(A11) $\Box\alpha \Rightarrow \Box\Box\alpha$,
(A12) $\Box 1$,

The rules of inference in L are Modus Ponens:

(MP): $\dfrac{\alpha,(\alpha \Rightarrow \beta)}{\beta}$

and necessitation rule :

$$\frac{\alpha \Rightarrow \beta}{\Box\alpha \Rightarrow \Box\beta}$$

Definition 10. [8] By a topological Heyting algebra we shall mean an algebra $(X;\rightarrow,\vee,\wedge,0,1)$ where $(X;\rightarrow,\vee,\wedge,1)$ is a Heyting algebra and moreover the following holds:

(TH1) $\Box x \wedge \Box y \leq \Box(x \wedge y)$,
(TH2) $\Box x \leq x$,
(TH3) $\Box x \leq \Box\Box x$,
(TH4) $\Box 1 = 1$,

for all $x, y \in X$.

In fact, corresponding to each S_{mBE}-system we can define the S_{mBE}-algebra in the obvious way. We now state the main theorem of this paper as follows:

Theorem 3. The class of all S_{mBE}-algebras coincides with the class of all topological Heyting algebras.

Proof. Let $\chi=(X;\rightarrow,\vee,\wedge,0,1)$ be in S_{mBE}-algebras. It follows that from (A1) -(A8), the algebraic properties corresponding to (A1) -(A8) are held.

Also, if $x \rightarrow y = 1$, and $y \rightarrow x = 1$, then $x = y$. Furthermore, we get that $x \rightarrow 1 = 1$.

Hence, by (TH1), (TH2) and Definition 4, we get that $(X;\to,1)$ is a positive implication algebra. Also, by (PI3)-(PI8) and Definition 5, $\chi=(X;\to,\vee,\wedge,0,1)$ is a Heyting algebra. Furthermore, the conditions (TH1)-(TH4) of Definition 10 are satisfied by (A9) -(A12). Thus the S_{mBE}-algebra $\chi=(X;\to,\vee,\wedge,0,1)$ is a topological Heyting algebra.

Conversely, if $\chi=(X;\to,\vee,\wedge,0,1)$ is a topological Heyting algebra, then by Definition 5, $(X;\to,1)$ is a positive implication algebra and the axioms (A3) -(A8) are satisfied. Also, the axioms (A1) and (A2) are satisfied by the equations (PI1), (PI2) of the Definition 4. The axioms (A9) -(A12) of S_{mBE}-algebra are satisfied by the axiom (TH1) -(TH4) of Definition 10, since $\chi=(X;\to,\vee,\wedge,0,1)$ is a topological Heyting algebra. Therefore $\chi=(X;\to,\vee,\wedge,0,1)$ is in S_{mBE}.

4. Conclusion

The S_{mBE}-algebras is constructed in Rasiowa's book style. Indeed, we showed that for a bounded commutative self-distributive BE-algebra, the class of all S_{mBE}-algebras, coincides with the class of all topological Heyting algebras.

There is an open problem: without above conditions on BE-algebra, does coincide with S_{mBE}-algebra?

In the future work we will answer to this problem.

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