

**Modules That Have a $\delta$–Supplement in Every Torsion Extension**

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(Received: 15.03.2016; Accepted: 06.06.2016)

**Abstract**

In this paper, we call a module $M$ a $\delta$-TE-module if $M$ has a $\delta$-supplement in every torsion extension. We obtain various properties of these modules. We show that every direct summand of a $\delta$-TE-module is a $\delta$-TE-module. We prove that the class of $\delta$-TE-modules is closed under extension under a special condition.

**Keywords** $\delta$–Supplement, Torsion Extension, $\delta$–Small

**Her Torsiyon Genişlemesinde Tümleyene Sahip Modüller**

**Özet**

Bu makalede her torsiyon genişlemesinde bir $\delta$–tümleyene sahip modül $M$ modülüne $\delta$-TE-modül diye isimlendirildi. Bu modüllerin birçok özellikleri elde edildi. $\delta$-TE-modüllerin direk toplamlarının da bir $\delta$-TE-modül olduğu gösterildi. $\delta$-TE-modül sınıfının özel bir koşulla genişlemeler altında kapalı olduğu ispatlandı.

**Anahtar Kelimeler** $\delta$–tümleyen, Torsiyon Genişlemesi, $\delta$–Küçük

1. **Introduction**

Throughout this paper, $R$ will be a commutative domain and all modules will be unital left $R$–modules, unless otherwise specified. Let $M$ be an $R$–module. By $N \leq M$ we mean that $N$ is a submodule of $M$. Recall that a submodule $N$ of $M$ is called small, denoted by $N \subseteq M$, if $N + L = M$ for all proper submodules $L$ of $M$. By $\text{Rad}(M)$, we denote the sum of all small submodule of $M$. Nevertheless a submodule $L$ of $M$ is said to be essential in $M$, denoted by $L \triangleleft M$, if $L \cap K \neq 0$ for each nonzero submodule $K$ of $M$. A module $M$ is said to be singular if $M \cong \frac{N}{L}$ for some module $N$ and a submodule $L$ of $N$ with $L \triangleleft N$.

Let $M$ and $N$ be $R$–modules. $N$ is called an extension of $M$ in case $M \subseteq N$. A module $M$ is said to be injective if it is a direct summand of every extension of itself [5].

As a proper generalization of direct summands of a module, one can define supplement submodules. The module $M$ is called supplemented, if every submodule $N$ of $M$ has a supplement in $M$, i.e. a submodule $K$ of $M$ minimal with respect to $M = N + K$. $K$ is a supplement of $N$ in $M$ if and only if $M = N + K$ and $N \cap K \triangleleft K$ [10].

As a generalization of small submodules, in [11] $\delta$–small submodules were introduced by Zhou. According to [11], a submodule $L$ of $M$ is called $\delta$–small in $M$, denoted by $L \subseteq \delta M$, if for any submodule $N$ of $M$ with $\frac{M}{N}$ singular, $M = N + L$ implies that $M = N$. The sum of all $\delta$–small submodules of a module $M$ is denoted by $\delta(M)$. It is easy to see that every small submodule of a module $M$ is $\delta$–small in $M$, so $\text{Rad}(M) \subseteq \delta(M)$ and
Rad(M) = δ(M) if M is singular. Also any non-singular semisimple submodule of M is δ-small in M and δ-small submodules of a singular module are small submodules. For more detailed discussion on δ-small submodules we refer to [11].

Let K, N be submodules of module M. N is called a δ-supplement of K in M, if M = N + K and N ∩ K ⊈ δ N. A module M is called δ-supplemented if every submodule of M has a δ-supplement in M [3,9]. On the other hand, a submodule N of M is said to have ample δ-supplements in M if every submodule L of M with M = N + L contains a δ-supplement of N in M. The module M is called amply δ-supplemented if every submodule of M has ample δ-supplements in M [7].

Let M be a module and N, K be any submodules of M with M = N + K. If N ∩ K ≤ δ(N) then N is called a generalized δ-supplement of K in M. Following [6], M is called a generalized δ-supplemented module (or briefly δ-GS module) if every submodule N of M has a generalized δ-supplement in M.

Modules that have supplements [ample supplements] in every module in which it is contained as a submodule have been studied in [12]. The structure of these modules have been determined over Dedekind domains. These modules are called modules with the property (E)[(EE)] in [12]. Such modules are also called supplementing modules in [1, p.255].

Let R be a commutative domain and M be an R-module. We denote by T(M), the set of all elements m of M for which there exists a non-zero element r of R such that rm = 0, i.e. Ann(m) ≠ 0. Then T(M), which is a submodule of M, called the torsion submodule of M. If M = T(M), then M is said to be a torsion module and M is torsion-free precisely when T(M) = 0 [5].

For modules M ⊆ N over a commutative domain, we say that N is a torsion extension of M if the factor module N/M is torsion. In a recent paper [2], modules that have a supplement in every torsion extension have been studied and these modules are called TE-modules. We call a module M δ-TE-module if M has a δ-supplement in every torsion extension. In this paper, we study some basic properties of these modules. We show how the class of δ-TE-modules is closed under direct summands, extensions and finite direct sums. We also prove that every submodule of a module is a δ-TE-module if and only if it has ample δ-supplements in every torsion extension.

2. Main Results

Proposition 2.1. Every direct summand of a δ-TE-module is a δ-TE-module.

Proof. Let M be a δ-TE-module and N be a direct summand of M. Then we can write M = N ⊕ K for some submodule K of M. For a torsion extension L of N, we denote by T the external direct sum L ⊕ K. Consider the canonical embedding ϕ: M → T. Then M ≅ ϕ(M) is a δ-TE-module and we have

\[
\frac{T}{ϕ(M)} = \frac{L ⊕ K}{ϕ(M)} ≅ \frac{L}{N}
\]

is torsion. Since φ(M) is a δ-TE-module, φ(M) has a δ-supplement U in T, that is, T = φ(M) + U and φ(M) ∩ U ⊈ δU. For the projection π : T → L, we have that L = π(U) + N. Also since Ker(π) ⊆ φ(M), we get

\[
π(φ(M) ∩ U) ⊆ π(φ(M)) ∩ π(U) = N ∩ π(U) ⊈ δπ(U)
\]

by [9, Lemma 1.3.(2)]. Hence, π(U) is a δ-supplement of N in L.
Proposition 2.2. Let $M$ be a module. Then the following statements are equivalent:

1. Every submodule of $M$ is a $\delta$–TE-module.
2. $M$ has ample $\delta$–supplements in every torsion extension.

Proof. (1)$\Rightarrow$(2) Suppose that every submodule of $M$ is a $\delta$–TE-module. For a torsion extension $N$ of $M$, let $N = M + K$ for some submodule $K$ of $N$. Note that $\frac{N}{M} \cong \frac{K}{M \cap K}$ is torsion.

Since $M \cap K$ is a $\delta$–TE-module, there exists a submodule $L$ of $K$ such that $K = (M \cap K) + L$ and $(M \cap K) \cap L = M \cap L \subseteq L$. Then we have $N = M + L$. Hence, $L$ is a $\delta$–supplement of $M$ in $N$.

(2)$\Rightarrow$(1) Let $T$ be any submodule of $M$. For a torsion extension $N$ of $T$, let $F = \frac{M \oplus N}{H}$, where the submodule $H$ is the set of all elements $(a, -a)$ of $F$ with $a \in T$ and let $\alpha : M \rightarrow F$ via $\alpha(m) = (m, 0) + H$, $\beta : N \rightarrow F$ via $\beta(n) = (0, n) + H$ for all $m \in M$, $n \in N$. It is clear that $\alpha$ and $\beta$ are monomorphisms. Thus we have the following pushout:

$$
\begin{array}{ccc}
T & \xrightarrow{\beta} & N \\
\downarrow{i_1} & & \downarrow{\beta} \\
M & \xrightarrow{\alpha} & F
\end{array}
$$

where $i_1$ and $i_2$ are inclusion mappings. It is easy to prove that $F = \text{Im}(\alpha) + \text{Im}(\beta)$.

Consider the epimorphism $\gamma : F \rightarrow \frac{N}{T}$ defined by $\gamma((m, n) + H) = n + T$ for all $(m, n) + H \in F$. Since $\ker(\gamma) = \text{Im}(\alpha)$, we have

$$\frac{N}{T} \cong \frac{F}{\text{Im}(\alpha)}$$

is torsion. By the hypothesis, $\text{Im}(\alpha)$ has ample $\delta$–supplements in every torsion extension because $\text{Im}(\alpha)$ is a monomorphism. Then, we can write

$$F = \text{Im}(\alpha) + V$$

and $\text{Im}(\alpha) \cap V \subseteq V$ with $V \subseteq \text{Im}(\beta)$. Hence we obtain that

$$N = \beta^{-1}(\text{Im}(\alpha)) + \beta^{-1}(V) = T + \beta^{-1}(V)$$

Suppose that $T \cap \beta^{-1}(V) + X = \beta^{-1}(V)$ for some submodule $X$ of $\beta^{-1}(V)$ with $\frac{\beta^{-1}(V)}{X}$ singular.

Then we have $V = V \cap \text{Im}(\beta) = \beta(\beta^{-1}(V)) = \beta(T \cap \beta^{-1}(V) + X)$

$$= \beta(T \cap \beta^{-1}(V)) + \beta(X) = \text{Im}(\alpha) \cap V + \beta(X).$$

Note that $\theta$ is an isomorphism. Hence

$$\frac{\beta^{-1}(V)}{X} \cong \frac{V}{\beta(X)}$$

is singular. Since $\text{Im}(\alpha) \cap V \ll_{\delta} V$, it follows that $\beta(X) = V$ and so that $X = \beta^{-1}(V)$. Thus $T \cap \beta^{-1}(V) \ll_{\delta} \beta^{-1}(V)$, that is, $\beta^{-1}(V)$ is a $\delta$–supplement of $T$ in $N$.

Theorem 2.1. Let $0 \rightarrow K \rightarrow M \rightarrow L \rightarrow 0$ be a short exact sequence. If $K$ and $L$ are $\delta$–TE modules with $L$ torsion, so does $M$.

Proof. Without loss of generality, we can assume that $K \leq M$ and $N$ be a torsion extension of $M$. For $K \leq M \leq N$, we have

$$\frac{N}{M} \cong \frac{K}{M} \cong \frac{K}{N}$$

is torsion and so $\frac{N}{K}$ is a torsion extension of $\frac{M}{K}$. Since $L \cong \frac{M}{K}$ is a $\delta$–TE-module, there
exists a submodule $V$ of $N$ such that

$$\frac{N}{K} = \frac{M}{K} + \frac{V}{K} \quad \text{and} \quad \frac{M \cap V}{K} \cong \frac{V}{K} \quad \text{and so}
$$

$$N = M + V. \text{ Since }$$

$$\frac{V}{K} \cong \frac{V}{M \cap V} \cong \frac{M + V}{M} = \frac{N}{M}.$$

$L$ is torsion, we obtain that $\frac{V}{K}$ is torsion. Then $K$ has a $\delta$-supplement $K_1$ in $V$, i.e. $V = K + K_1$ and $K \cap K_1 \subseteq \delta K_1$ because $K$ is a $\delta$-TE-module. Therefore $N = M + X = M + K_1$. Assume that $N = M + X$ for some submodule $X$ of $K_1$. Then

$$\frac{M + X + K}{K} = \frac{N}{K}.$$

Note that

$$\frac{K_1}{X} \cong \frac{K_1 + K}{X + K} = \frac{V}{X + K} = \frac{V}{X + K}.$$

is singular. It follows from [4, Lemma 2.1], $\frac{V}{K} = \frac{X + K}{K}$ and so $V = X + K$. Since $K_1$ is a $\delta$-supplement of $K$ in $V$, by [4, Lemma 2.1], by we have that $X = K_1$. Thus $K_1$ is a $\delta$-supplement of $M$ in $N$. Thus $K_1$ is a $\delta$-supplement of $M$ in $N$.

**Corollary 2.1.** Let $M_1$ and $M_2$ be $\delta$-TE-modules with $M_2$ torsion and $M = M_1 \oplus M_2$. Then $M$ is a $\delta$-TE-module.

**Proof.** Let $M = M_1 \oplus M_2$. By using the following short exact sequence

$$0 \to M_1 \to M \to M_2 \to 0$$

we obtain that $M$ is a $\delta$-TE-module by Theorem 2.1.

**Lemma 2.1.** Let $M$ be a $\delta$-TE-module and $N$ be a torsion extension of $M$ such that $\delta(N) = 0$. Then $M$ is a direct summand of $N$.

**Proof.** By assumption, $M$ has a $\delta$-supplement in $N$, say $K$. Since $M \cap K \subseteq \delta K$, it follows that $M \cap K \subseteq \delta(K) = 0$. Hence $N = M \oplus K$.

In [8], a ring $R$ is called a left $\delta$-V-ring, if for any left $R$-module $M$, $\delta(M) = 0$.

**Corollary 2.2.** Let $M$ be a $\delta$-TE-module over a $\delta$-V-ring. Then $M$ is a direct summand of any module $N$ with $\frac{N}{M}$ torsion.

3. References